where $K(t',t'')$ is the controllability matrix associated with the time interval $[t',t'']$ as defined in Eq. (3). By assumption $K(t',t'')$ is nonsingular. Hence Eq. (13) implies $\lambda^*(t')=0$. But the initial conditions $x^*(t')=0$, $\lambda^*(t')=0$ for the state/adjoint system [Eq. (8)] in conjunction with the assumed continuity of all of the participating matrix functions of time A, B, Q, and R on $[t_c-\Delta, t_c)\cup(t_c, t_c+\delta]$ immediately imply that $x^*(t)\equiv 0$, $\lambda^*(t)\equiv 0$ on $[t_c-\Delta, t_c+\delta]$, even if A, B, Q, and R are not continuous across t_c. But this contradicts $x^*(t_c+\delta)=\hat{x}_F\neq 0$. Hence path 2 is not optimal. Q.E.D.

Summary

A new proof is presented for Jacobi's no-conjugate-point necessary condition. This is achieved by deriving a no-conjugate-point condition for a certain class of LQP. Through the concept of the accessory minimum problem, the result can be generalized to nonlinear optimal control problems. In contrast to earlier results, the new proof also applies if the coefficient functions of time associated with the accessory minimum problem have any finite number of discontinuities.

Acknowledgments

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Riccati Solution for the Minimum Model Error Algorithm

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Introduction

CONVENTIONAL filter/smoother algorithms, such as the Kalman filter,[1] require detailed knowledge of both the model error and the measurement error. In nearly all circumstances, the measurement error characteristics are known a priori; however, the errors in the actual system are generally unknown. Also, the Kalman filter assumes that the model error is modeled by a Gaussian noise process. In many cases, such as nonlinearities in the actual system response, this assumption can lead to severely degraded state estimates. The minimum model error[2] (MME) algorithm provides a method of determining optimal state estimation in the presence of significant error in the assumed (nominal) model. The advantages of this method are: 1) the model error and process noise are assumed unknown and are estimated as part of the solution, 2) the model error may take any form (even nonlinear), and 3) the algorithm is robust in the presence of high noise measurements. In several previous studies, this algorithm has been successfully applied to numerous applications, including nonlinear estimation[3] and robust realization/identification of mode shapes in damped structures.[4,5]

The determination of the optimal state estimates using the MME algorithm is derived from a minimization of a cost functional subject to differential equation constraints. In this Note, a closed-form solution to the two-point boundary value problem (TPBVP), associated with the MME algorithm, is developed for linear time-variant state-space models. The closed-form solution is derived using an inhomogeneous Riccati transformation. The resulting forms include one nonlinear Riccati equation and one linear differential equation, each with discrete updates at every output measurement interval.

Minimum Model Error Algorithm

In this section, the MME algorithm is briefly reviewed for the case of linear time-variant state-space models. A more detailed derivation of the algorithm may be found in Ref. 2. The MME algorithm assumes that the state estimates are given by a nominal (prespecified) model and an unmodeled disturbance vector, shown as

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t) + d(t)\\ y(t) &= C(t)x(t) + D(t)u(t)\end{aligned} \tag{1}$$

where $A(t)$, $B(t)$, $C(t)$, and $D(t)$ are time-variant nominal state matrices, $u(t)$ is a known forcing input, $d(t)$ is an $(n\times 1)$ unmodeled (to-be-determined) error vector, $x(t)$ is the $(n\times 1)$ state estimate vector, and $y(t)$ is the $(m\times 1)$ estimated output. For the remainder of this Note, the state-space (model) matrices are assumed time variant but are shown without the time argument (t).

State-observable discrete time-domain measurements are assumed for Eq. (1) in the following form:

$$\tilde{y}(t_k) = g_k\left[x(t_k), t_k\right] + v_k \tag{2}$$

where $\tilde{y}(t_k)$ is an $(m\times 1)$ measurement vector at time t_k, g_k is an accurate model of the measurement process, v_k represents measurement noise, and m is the total number of measurement output sets. The measurement noise process is assumed to be a zero-mean, Gaussian distributed process of known covariance R.

In the MME, the optimal state estimates are determined on the basis that the measurement-minus-estimate error covariance matrix must match the measurement-minus-truth error covariance matrix. This condition is referred to as the covariance constraint, approximated by

$$\left\{\left[\tilde{y}(t_k)-y(t_k)\right]\left[\tilde{y}(t_k)-y(t_k)\right]^T\right\} \approx R \tag{3}$$

Therefore, the estimated measurements are required to fit the actual measurements with approximately the same error covariance as the actual measurements fit the truth.

A cost functional, consisting of the weighted sum square of the measurement-minus-estimate residuals plus the weighted sum square of the model correction term, is next minimized

$$J = \sum_{k=1}^{m}\left\{\left[\tilde{y}(t_k)-y(t_k)\right]^T R^{-1}\left[\tilde{y}(t_k)-y(t_k)\right]\right\} + \int_{t_0}^{t_f} d(\tau)^T W d(\tau)\,\mathrm{d}\tau \tag{4}$$

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where W is a weight matrix determined by satisfying the covariance constraint. If the measurement residual covariance is larger than R, then the measurement estimate is not close to the actual system measurements. Therefore, W should be decreased to less penalize the model correction $[d(t)]$. However, if the estimate covariance is too low, then W should be increased to allow more model correction.

The necessary conditions for the minimization of J, with respect to the model correction term $d(t)$, leads to the following TPBVP:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + d(t) \\ \dot{\lambda}(t) &= -A^T\lambda(t) \\ d(t) &= -\tfrac{1}{2}W^{-1}\lambda(t) \\ \lambda(t_k^+) &= \lambda(t_k^-) + 2C^TR^{-1}\left[\tilde{y}(t_k) - y(t_k)\right] \end{aligned} \tag{5}$$

where $\lambda(t)$ is a vector of costates (Lagrange multipliers). Also, the costate equation is updated at each measurement interval. The boundary conditions are selected such that either $\lambda(t_0^-) = \mathbf{0}$ or $x(t_0)$ is specified for the initial time and either $\lambda(t_f^+) = \mathbf{0}$ or $x(t_f)$ is specified for the final time.

Sweep Solution

Riccati solutions are useful in determining optimal trajectories for the design of linear control systems with quadratic cost functionals. The application of this method is expanded to derive matrix Riccati solutions for the MME algorithm, which includes discrete updates in the costate equation. The costate equation is first assumed to be a linear function with respect to the state estimates

$$\lambda(t) = K(t)x(t) + h(t) \tag{6}$$

where $K(t)$ is an $(n \times n)$ homogeneous operator and $h(t)$ is an $(n \times 1)$ inhomogeneous operator. By differentiating Eq. (6) with respect to time and substituting into Eq. (5), the following equations are developed:

$$\begin{aligned} \dot{K}(t) &= -K(t)A + \tfrac{1}{2}K(t)W^{-1}K(t) - A^TK(t) \\ \dot{h}(t) &= \left[\tfrac{1}{2}K(t)W^{-1} - A^T\right]h(t) - K(t)Bu(t) \end{aligned} \tag{7}$$

The update equations are also derived by substituting Eq. (7) into the update costate equation in Eq. (5). Realizing that the coefficient for the state trajectory must vanish yields the two update equations

$$\begin{aligned} K(t_k^-) &= K(t_k^+) + 2C^TR^{-1}C \\ h(t_k^-) &= h(t_k^+) + 2C^TR^{-1}\left[Du(t_k) - \tilde{y}(t_k)\right] \end{aligned} \tag{8}$$

with boundary conditions of $K(t_f^+) = 0$, $h(t_f^+) = \mathbf{0}$, and $x(t_0)$ specified. The discrete update equation for the first Riccati solution has a constant update at each measurement interval. However, as shown later, the solution to its corresponding Riccati equation does indeed reach steady-state values for time-invariant state-space model matrices.

The solution for the optimal state estimates is first derived by integrating Eq. (7) backward, with the discrete updates at each measurement point. The Riccati trajectories are then stored, and the optimal state equations are integrated forward

$$\dot{x}(t) = \left[A - \tfrac{1}{2}W^{-1}K(t)\right]x(t) - \tfrac{1}{2}W^{-1}h(t) + Bu(t) \tag{9}$$

From this equation, the homogeneous Riccati trajectory along with the weighting matrix act to correct the nominal state matrix, and the inhomogeneous Riccati trajectory acts as another forcing input into the system. For an illustration of the usefulness of the simple Riccati solution, an example is shown in the next section.

Numerical Example

A sample measurement trajectory, with white mean Gaussian noise added to the true measurements, of 101 points is generated using the following "truth" transfer function:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [45 \quad 2] \quad D = 0 \tag{10}$$

The assumed model has repeated eigenvalues with no numerator dynamics, and the state-space realization is given as

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [1 \quad 0] \quad D = 0 \tag{11}$$

With a given W, the first step is to integrate the matrix Riccati equations (7) backward to the initial time, accounting for the jump discontinuities, given by Eq. (8), at each measurement time. Then, using the stored homogeneous and inhomogeneous Riccati trajectories, the state trajectory is integrated forward to the final time. Finally, the covariance constraint is checked, and W is adjusted to compensate for the error in the state estimates.

Figure 1 shows the noisy measurements along with the output estimates and true output time history. The program is executed in double precision FORTRAN. The MME algorithm using the sweep solution provides optimal state estimates with an accuracy less than 1% to the actual true trajectory.

Figure 2 shows the homogeneous solution to the Riccati equation given in Eq. (7). For the case of time-invariant state-space models, this Riccati equation does reach a steady-state value for each matrix component. Therefore, an algebraic Ric-

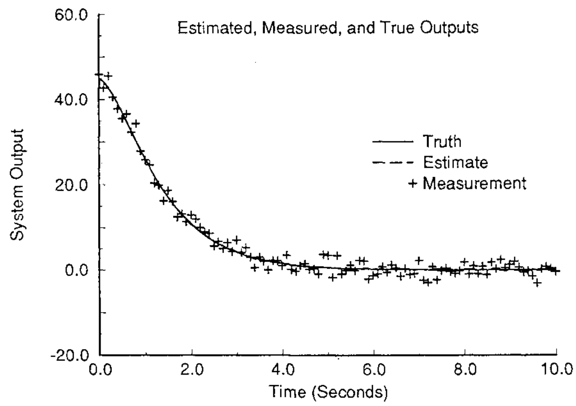

Fig. 1 Optimal output estimates and measurements.

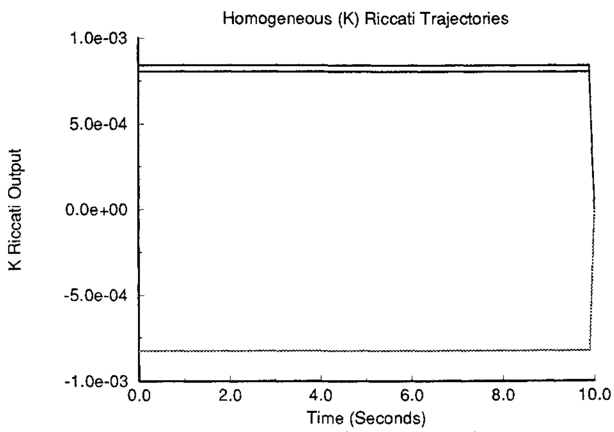

Fig. 2 Homogeneous Riccati trajectories.

cati equation can be used, which greatly reduces the computation burden. Also, not shown here, the linear differential equation in Eq. (7) "tracks" the actual state trajectories and serves to force the nominal-model state trajectories to the actual state trajectories.

Finally, if W is chosen too small from the optimal weighting matrix, the resulting matrix Riccati equations are "stiff" and tend to diverge very quickly. This can be extremely useful in determining a good starting guess for the weighting matrix.

Conclusions

This Note has established a matrix Riccati solution for the minimum model error estimation algorithm. The functional form for the solution of the associated two-point boundary value problem includes one Riccati equation and one linear differential equation with discrete update discontinuities at each measurement time. The homogeneous Riccati equation can be reduced to an algebraic Riccati equation if the assumed model is linear and time-invariant, thus reducing the solution to linear differential and algebraic equations. The algorithm was demonstrated for a simple linear time-invariant model. Results indicate that the algorithm is capable of determining optimal state estimates by using the closed-form solution.

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Optimal Control System Design with Prescribed Damping and Stability Characteristics

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Introduction

DESIGN of multivariable optimal control systems with specified damping and stability characteristics has been a goal pursued by a number of researchers. Practical time-domain performance requirements often include the following two important specifications: 1) the response must be sufficiently fast and smooth, and 2) the response must not exhibit excessive overshoot and oscillations. The first specification places a bound on the settling time, whereas the second one gives rise to a bound on the damping ratio. For this reason, the shaded area of Fig. 1 has been widely accepted as a suitable design sector. Because direct optimal pole placement in the shaded area is a very difficult problem to solve, a multitude of approximate regions have been proposed in the literature.[1-5] The following theorem has been crucial in developing most of the root-clustering algorithms.

Relative Stability Theorem: The eigenvalues of the matrix A lie within the shaded stability region of Fig. 2, if and only if the eigenvalues of the $2N \times 2N$ matrix

$$A_\alpha \otimes \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \tag{1}$$

have negative real parts. The angle β is given by $\beta = \pi/2 - \theta$, and A_α is defined as $A - \alpha I$. See Ref. 6 for the proof.

For example, a design procedure based on the preceding theorem recently appeared in Ref. 3. The straightforward implementation of this theorem, however, results in a root-clustering sector that underestimates the design region as can be seen in Fig. 2. Hence, the main goal of this Note is to develop a new design method based on a more accurate approximation of the shaded region of Fig. 1.

Problem Formulation and Solution

Consider the following linear time-invariant dynamic system

$$\dot{X} = AX + BU \tag{2}$$

where A and B are constant matrices of order $N \times N$ and $N \times M$, respectively. The problem solved in this Note is to determine a state feedback controller of the form $U = KX$ such that 1) the closed-loop system matrix $A + BK$ has all of its eigenvalues within the shaded region of Fig. 1, and 2) the following linear quadratic performance index is minimum

$$J = \int_0^\infty (X^TQX + 2X^TMU + U^TRU)\,dt$$

The preceding eigenvalue-clustering problem is solved by transforming the shaded area into the left-hand plane of an associated dynamic system. To begin with the transformation process, consider the Laplace transform of Eq. (2), assuming zero initial conditions:

$$sX(s) = AX(s) + BU(s) \tag{3}$$

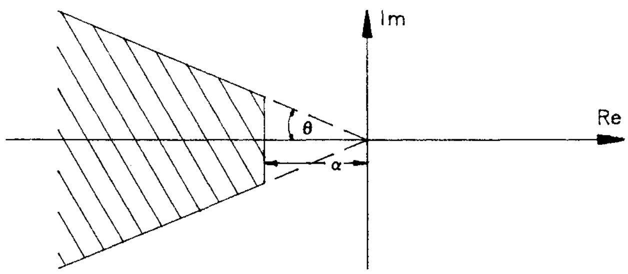

Fig. 1 Design sector with prescribed damping and stability specifications.

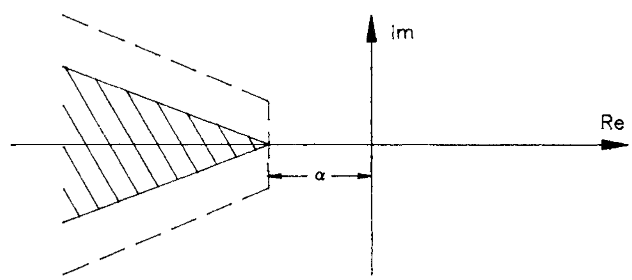

Fig. 2 Design sector resulting from direct implementation of the relative stability theorem.

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